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# ON UNIT GROUPS AND CLASS GROUPS OF QUARTIC FIELDS OF SIGNATURE (2, 1)

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ABSTRACT. This is the third and last paper of a series, now completing the description of the unit group and class group of all quartic number fields $F$ of discriminant $d_F$ with $|d_F| < 10^6$.

## 1. INTRODUCTION

In this paper we consider quartic fields $F$ with two real conjugates. Using the tables of David Ford and the first two authors [2], we computed unit groups $U_F = \langle -1 \rangle \times \langle \varepsilon_1 \rangle \times \langle \varepsilon_2 \rangle$ and class groups $Cl_F$ of all 90671 number fields $F$ whose discriminant $d_F$ is bounded in absolute value by one million. A comparison shows that our results are not in precise agreement with the predictions of Cohen and Martinet [4]. However, this was not to be expected because of the relatively small range of discriminants under consideration.

After [3] and [5], this paper completes the description of the most important invariants of all quartic number fields $F$ with $|d_F| < 10^6$.

## 2. UNIT GROUPS

The fundamental units were computed by using the generalized Voronoi algorithm [1]. The algorithm operates as follows: For $\alpha \in F$, let $\alpha^{(1)}, \alpha^{(2)}$ be the real conjugates of $\alpha$, and $\alpha^{(3)}, \alpha^{(4)}$ the nonreal, complex conjugates; i.e., $\alpha^{(4)}$ is complex conjugate to $\alpha^{(3)}$. We set

$$|\alpha|_1 := |\alpha^{(1)}|,$$
$$|\alpha|_2 := |\alpha^{(2)}|,$$
$$|\alpha|_3 := |\alpha^{(3)}|^2.$$

A fractional ideal $\mathbf{a}$ in $F$ is called *reduced* if

$$1 \in \mathbf{a} \quad \text{and} \quad \{\alpha \in \mathbf{a} : |\alpha|_i < 1,\ 1 \le i \le 3\} = \{0\}.$$

For $i \in \{1, 2, 3\}$ the $i$-neighbor of a reduced ideal $\mathbf{a}$ is defined as the ideal $\mathbf{b} = \frac{1}{\alpha}\mathbf{a}$ for $\alpha \in \mathbf{a}$ subject to

$$|\alpha|_i > 1, \qquad |\alpha|_j < 1 \quad \text{for } j \in \{1, 2, 3\} \setminus \{i\}$$

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and

$$\{\alpha' \in \mathbf{a} : |\alpha'|_\iota < \max\{1, |\alpha|_\iota\}\} = \{0\}.$$

In [1] it is proved that a system of fundamental units can be obtained as follows:

**Algorithm.**
*Input*: maximal order $O_F$ of $F$.
*Output*: system $\{\varepsilon_1, \varepsilon_2\}$ of fundamental units.
found = false, $k = 1$, $\mathbf{a}_1 = o_F$.
**While** not found **do**
{
Compute 1-neighbor $\mathbf{a}_{k+1} = \frac{1}{\alpha_k}\mathbf{a}_k$ of $\mathbf{a}_k$.
**If** there is $l_1 < k$ with $\mathbf{a}_{l_1} = \mathbf{a}_{k+1}$
**then** found = true,
**else** $k = k + 1$.
}
found = false, $i = 1$, $\mathbf{b}_1 = \mathbf{a}_{l_1}$.
**While** not found **do**
{
Compute 2-neighbor $\mathbf{b}_{i+1} = \frac{1}{\beta_i}\mathbf{b}_i$ of $\mathbf{b}_i$.
**If** there is $l_2 \in \{l_1, \dots, k\}$ with $\mathbf{a}_{l_2} = \mathbf{b}_{i+1}$
**then** found = true,
**else** $i = i + 1$.
}
Set

$$\varepsilon_1 = \prod_{j=l_1}^{k} \alpha_j \quad \text{and} \quad \varepsilon_2 = \prod_{j=1}^{i} \beta_j / \prod_{j=l_1}^{l_2-1} \alpha_j.$$

**End.**

In the following table we show the magnitudes of the regulators

$$R_F := \left| \det \begin{pmatrix} \log|\varepsilon_1^{(1)}| & \log|\varepsilon_2^{(1)}| \\ \log|\varepsilon_1^{(2)}| & \log|\varepsilon_2^{(2)}| \end{pmatrix} \right|$$

in dependence on the Galois group structure:

| | D4 | | S4 | | #/frequency | |
|---|---|---|---|---|---|---|
| #/frequency | 9772 | 10.78% | 80899 | 89.22% | 90671 | 100% |
| $0 < R_F < 1$ | 6 | 0.06% | 9 | 0.01% | 15 | 0.02% |
| $1 \leq R_F < 5$ | 485 | 4.96% | 347 | 0.43% | 832 | 0.92% |
| $5 \leq R_F < 10$ | 1438 | 14.72% | 1719 | 2.12% | 3157 | 3.48% |
| $10 \leq R_F < 20$ | 2624 | 26.85% | 6430 | 7.95% | 9054 | 9.99% |
| $20 \leq R_F < 50$ | 3428 | 35.08% | 23943 | 29.60% | 27371 | 30.19% |
| $50 \leq R_F$ | 1791 | 18.33% | 48451 | 59.89% | 50242 | 55.41% |

## 3. Class groups

Since for the class group computation the same algorithm as in [3, 5] was used, we will not present details. The main idea is to compute all prime ideals with norm below the Zimmert bound [9]

$$M_F = \left\lfloor \frac{1}{6.792} \sqrt{-d_F} \right\rfloor$$

and to compute sufficiently many relations between those ideals (see [6, 8]).

We first show the distribution of noncyclic class groups:

| D4 | S4 | $\sum$ |
|---|---|---|
| 244 | 125 | 369 |
| 0.25% | 0.15% | 0.41% |

The following table shows the distribution of class numbers:

| | D4 | | S4 | | #/frequency | |
|---|---|---|---|---|---|---|
| #/frequency | 9772 | 10.78% | 80899 | 89.22% | 90671 | 100% |
| $h_F = 1$ | 5199 | 53.20% | 68533 | 84.71% | 73732 | 81.32% |
| $h_F = 2$ | 2839 | 29.05% | 9270 | 11.46% | 12109 | 13.35% |
| $h_F = 3$ | 501 | 5.13% | 1462 | 1.81% | 1963 | 2.16% |
| $h_F = 4$ | 769 | 7.87% | 1037 | 1.28% | 1806 | 1.99% |
| $5 \le h_F < 10$ | 420 | 4.30% | 574 | 0.71% | 994 | 1.10% |
| $10 \le h_F < 20$ | 42 | 0.43% | 23 | 0.03% | 65 | 0.07% |
| $20 \le h_F$ | 2 | 0.02% | 0 | 0.00% | 2 | 0.00% |

We finally present the frequency of each class group structure and the corresponding minimal field discriminant (if greater than $-10^6$):

| $h_F$ | $Cl_F$ | D4 | S4 | # |
|---|---|---|---|---|
| 1 | 1 | 5199 (−275) | 68533 (−283) | 73732 |
| 2 | 2 | 2839 (−7975) | 9270 (−6848) | 12109 |
| 3 | 3 | 501 (−20975) | 1462 (−25471) | 1963 |
| 4 | 4 | 547 (−51207) | 916 (−54764) | 1463 |
| 4 | 2, 2 | 222 (−83600) | 121 (−132800) | 343 |
| 5 | 5 | 135 (−82975) | 311 (−69128) | 446 |
| 6 | 6 | 154 (−190400) | 110 (−137300) | 264 |
| 7 | 7 | 48 (−165744) | 75 (−169924) | 123 |
| 8 | 8 | 47 (−218975) | 54 (−273491) | 101 |
| 8 | 2, 4 | 21 (−319424) | 4 (−804875) | 25 |
| 9 | 9 | 15 (−323975) | 20 (−326111) | 35 |
| 10 | 10 | 14 (−451975) | 9 (−645700) | 23 |
| 11 | 11 | 9 (−593856) | 14 (−436227) | 23 |
| 12 | 12 | 11 (−367975) | — | 11 |
| 12 | 2, 6 | 1 (−995600) | — | 1 |
| 13 | 13 | 3 (−645056) | — | 3 |
| 14 | 14 | 2 (−788975) | — | 2 |
| 16 | 16 | 2 (−328975) | — | 2 |
| 20 | 20 | 1 (−302975) | — | 1 |
| 23 | 23 | 1 (−616475) | — | 1 |
| # | | 9772 | 80899 | 90671 |

The computations were done on Apollo workstations (CPU Motorola 68030), using the number-theoretic program package KANT (see [7]). All data can be obtained from the second author.

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